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The cosmological constant problem from a brane-world perspective

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Abstract: We point out several subtleties arising in brane-world scenarios of cosmological constant cancellation. We show that solutions with curvature singularities are inconsistent, unless the contribution to the effective four-dimentional cosmological constant of the physics that resolves the singularities is fine-tuned. This holds for both flat and curved branes. Irrespective of this problem, we then study an isolated class of flat solutions in models where a bulk scalar field with a vanishing potential couples to a 3-brane. We give an example where the introduction of a bulk scalar potential results in a non-zero cosmological constant. Finally we comment on the stability of classical solutions of the brane system with respect to quantum corrections.

Keywords: Field Theories in Higher Dimensions, Extra Large Dimensions, p-branes, Cosmology of Theories beyond the SM.

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1. Introduction

With the discovery of string dualities it became clear that extended objects (branes) are built into any string theory. In the form of D-branes [1] they first served as tools in addressing conceptual questions of string theory (e.g. dualities, black hole entropy, holography). An interesting property of D-branes is that they provide a natural picture for having some fields (typically gauge fields) being confined to a hypersurface in space whereas others (typically gravitational fields) can propagate in all directions. This property makes contact to an early suggestion [2, 3] that the matter in our universe is confined to live on a $(3+1)$-dimensional hypersurface of a higher-dimensional universe. In the simplest case the geometry is such that the higher-dimensional space is a direct product of our $(3+1)$-dimensional universe with some additional internal space. This picture was generalized to warped compactifications [4], where the $(3+1)$-dimensional metric depends on the position of the brane in the higher-dimensional space. The authors of [4] constructed a model where this dependence is exponential. They were led to the conjecture that a natural explanation for the large hierarchy between the electroweak scale and the Planck scale was found. But even without solving the hierarchy problem warped compactifications open up interesting possibilities, like localizing gravity on the brane with an infinitely large extra dimension [5].

Apart from the hierarchy problem it has been also tried to solve the cosmological constant problem[1] within the brane world scenario [9, 10]. The basic new ingredient is that there is a scalar living in the bulk with coupling to the brane. Naively one

[1]For reviews of the cosmological constant problem see [6, 7, 8].

then hopes that the well-known fine-tuning of the parameters of the theory required in order to ensure a vanishing cosmological constant is replaced by the adjustement of the zero mode of this additional scalar. However, these models typically have singularities being located within a finite distance from the brane. Problems arising due to the singularities have for example been pointed out in [11, 12] (this and other aspects of these models have been also the subject of refs. [13]–[19]). In our note [12] we observed that the singularities can be given a physical interpretation in the presence of additional source terms. Then the curvature singularity just reflects the fact that we are considering the limit in which the sources (or branes) at the singularities have no finite extension into the transverse direction (this is similar to black hole singularities appearing due to the point like nature of the source). With these additional sources, however, a fine-tuning is needed to obtain a vanishing cosmological constant for the effective four-dimensional theory. Irrespective of this problem, it has been stressed in [9, 20] that for a very particular choice of parameters (namely vanishing bulk scalar potential and particular form and value of the scalar-brane coupling) there exist isolated flat solutions, i.e. there seems to be no smooth deformation connecting zero cosmological constant to a non-vanishing value.

In the present paper we are going to elaborate on brane world models as a framework to find solutions of the cosmological constant problem. In particular we want to clarify the following points:

- the vanishing of the cosmological constant requires a consistency condition for the brane-world setup which is especially crucial in the presence of singularities (this has been sometimes overlooked in the literature);

- known mechanisms to fullfill this consitency condition require a fine tuning of parameters of the model, comparable to the usual fine tuning of the cosmological constant.

The so-called self tuning solution of the cosmological constant problem is therefore at best a scenario where this problem has been rephrased. The real solution of the problem would have to explain the fine tuning that is necessary for the consistency condition mention above.

The paper is organized as follows. In the next section we specify the general setup and recall consistency conditions on warped compactifications. In section three we generalize the points made in [12]. We show in various examples (flat and curved branes) that one can make solutions with singularities consistent by adding additional branes to the model. To achieve a definite value for the cosmological constant (e.g. zero) a fine-tuning is needed. Section 4 is devoted to a careful investigation of the isolated flat solution mentioned above, for which we find that introducing a non-trivial bulk potential for the scalar field necessarily results in a non-zero cosmological constant. In section 5 we discuss the relation of the classical solutions of the brane-bulk system to the physical quantum world. Finally section 6 is devoted to a summary and outlook.

2. General setup

To study thoroughly the structure of vacuum solutions in the brane worlds it is useful to consider an ansatz corresponding to maximally symmetric 4d foliations, which beyond the Minkowski space include de Sitter and Anti-de Sitter 4d space-times. The simplest non-trivial setup corresponds to just a single transverse coordinate. The line element is

$$ds^2 = e^{2A(x^5)}\tilde{g}_{\mu\nu}dx^\mu dx^\nu + \left(dx^5\right)^2 \tag{2.1}$$

with $\tilde{g}_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ where the background metric corresponds to one of the 4d maximally symmetric spaces: $\bar{g}_{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$ (Minkowski metric); $\mathrm{diag}(-1, e^{2\sqrt{\bar{\Lambda}}t}, e^{2\sqrt{\bar{\Lambda}}t}, e^{2\sqrt{\bar{\Lambda}}t})$ (dS_4 background with $\bar{\Lambda} > 0$); $\mathrm{diag}(-e^{2\sqrt{-\bar{\Lambda}}x^3}, e^{2\sqrt{-\bar{\Lambda}}x^3}, e^{2\sqrt{-\bar{\Lambda}}x^3}, 1)$ (AdS_4 background, $\bar{\Lambda} < 0$). With $h_{\mu\nu}$ we denote a small fluctuation around the background metric. The sources for such configurations are assumed to be located on a number of four-dimensional branes, one of which represents the observable gauge sector. It is important to note that although the observable gauge interactions are strictly confined to the 3-brane, the gravity and moduli fields permeate the whole space, effectively connecting the walls in a non-trivial way. This implies that every particle localized on the wall feels sources of gravitational forces which are located all over the bulk. In some cases the influence of the remote sources will be suppressed, like in the case of the exponentially falling off graviton wave function in ref. [5], which would effectively restrict the relevant gravitational sources to the thin layer around the brane, but sometimes the suppression would be so mild, that the influence of the whole bulk contribution will be highly relevant. Thus in any case, even though the gauge forces are restricted to the branes, the gravitational sector has to be completely integrated out when going to the effective four-dimensional theory. In particular, this implies that when one computes the effective four-dimensional energy density, or four-dimensional vacuum pressure, one has to integrate over the whole causally accessible portion of the transverse space.

Let us analyze the relation between 4d and 5d physics in some detail. The 5d action we consider is (we follow the conventions of ref. [10] for the normalization of the Einstein term and of the scalar kinetic term)

$$\begin{aligned} S_5 = &\int d^5x\sqrt{-G}\left(R - \frac{4}{3}\left(\partial\phi\right)^2 - V\left(\phi\right)\right) + \\ &+ \int_{M_i} d^5x\sqrt{-g}\left(-f_i\left(\phi\right)\delta\left(x^5 - x_i^5\right)\right), \end{aligned} \tag{2.2}$$

where $V(\phi)$ is the bulk scalar potential, $f_i(\phi)$ are brane contributions to the lagrangian, the index i counts the branes, and $g_{\mu\nu}$ is the induced metric on the branes. The corresponding Einstein equations are

$$\sqrt{-G}\left(E_{MN}-\frac{1}{2}T_{MN}\right)=0\,, \tag{2.3}$$

where $E_{MN}=R_{MN}-\frac{1}{2}G_{MN}R$ is the Einstein tensor and

$$T_{MN} \;=\; \frac{8}{3}\partial_M\phi\partial_N\phi-\frac{4}{3}(\partial\phi)^2G_{MN}-V(\phi)G_{MN}-f_i\delta\left(x^5-x_i^5\right)g_{\mu\nu}\delta^\mu_M\delta^\nu_N \tag{2.4}$$

is the energy-momentum tensor. The dilaton equation of motion is

$$-\frac{\partial V}{\partial\phi}\sqrt{-G}-\sqrt{-g}\,\delta\left(x^5-x_i^5\right)\frac{\partial f_i}{\partial\phi}+\frac{8}{3}\partial_M\left(\sqrt{-G}G^{MN}\partial_N\phi\right)=0\,, \tag{2.5}$$

where $M,N=1,\ldots,5$ and $\mu,\nu=1,\ldots,4$. Although we do not consider fluctuations of G_{55} that would correspond in four dimensions to the modulus associated with the length of the fifth dimension and put here $G_{55}=1$, we shall comment on the stability of the fifth dimension in the case of static solutions at the end of the paper. With the choice $\bar{g}_{\mu\nu}=\eta_{\mu\nu}$, the ansatz (2.1) naively implies that the 4d cosmological constant should vanish. As explained in [12] this is the case if the ansatz solves Einstein equations globally, i.e. everywhere in 5d space-time. If however there happens to be singularities, for instance at points where $e^{A(x^5)}=0$, then one needs to "repair" the system at such points in order to achieve truly flat 4d background. This issue shall be discussed later on. In the presence of the general sources the warp factor, hence also the curvature scalar, as well as the bulk scalar ϕ do assume some non-trivial and x^5-dependent vacuum configuration.

To identify 4d gravity and the strenght of its coupling to matter one needs to split the action (2.2) into vacuum part, and fluctuations around it. The vacuum part is read off from the 5d action upon substituting solutions of the equations of motion for the warp factor and for the scalar. This vacuum lagrangian is

$$\langle L_5\rangle=\int dx^5\,e^{4\langle A(x^5)\rangle}\left(\frac{2}{3}V\left(\langle\phi\rangle\right)+\frac{1}{3}\langle f_i\rangle\delta\left(x^5-x_i^5\right)\right), \tag{2.6}$$

where $\langle\cdots\rangle$ indicates that the classical background value of the corresponding field is taken. It is immediate to see that the lowest terms in $h_{\mu\nu}$ of the expansion of the action (2.2) around the vacuum are

$$S_5=\int d^5x\sqrt{-\tilde{g}}M^3e^{2A(x^5)}\tilde{R}+\cdots\,, \tag{2.7}$$

where $\tilde{R}$ denotes the curvature built out of $\tilde{g}_{\mu\nu}$. From (2.7) one can identify the 4d Planck scale $M_{\rm Pl}^2=M^3\int dx^5e^{2\langle A(x^5)\rangle}$. To obtain the effective four-dimensional action S_4 for gravity one needs to integrate over the fifth coordinate. The model is consistent if $\langle \tilde{g}_{\mu\nu} \rangle$ from our ansatz (2.1) minimizes S_4. Thus the effective four-dimensional action reads

$$S_4 = M_{\rm Pl}^2 \int d^4x \sqrt{-\tilde{g}}\, (\tilde{R} - \lambda), \tag{2.8}$$

where the cosmological constant $\lambda = 6\bar{\Lambda}$. Finally we arrive at a consistency condition by the requirement that the on-shell values of S_5 and S_4 should be equal[2]

$$\langle L_5 \rangle \;=\; M_{\rm Pl}^2 \left(\langle \tilde{R} \rangle - \lambda \right) \;=\; 6\bar{\Lambda} M_{\rm Pl}^2 \,. \tag{2.9}$$

A breakdown of this condition would signal that Einstein equations are not satisfied everywhere in space-time. It is interesting to note that eq. (2.9) can be rewritten

$$-\frac{1}{3} \int dx^5 \, e^{4\langle A \rangle} \, \langle T_0^0 + T_5^5 \rangle \;=\; 6\bar{\Lambda} M_{\rm Pl}^2 \,. \tag{2.10}$$

In the case of a Poincaré-invariant background, the vanishing of the 4d cosmological constant is therefore equivalent to a constraint on the 5d energy-momentum tensor, namely the integral on the left-hand side of (2.10) should vanish. This constraint is a variant of the condition $\int dx^5 e^{\langle A \rangle} \langle T_0^0 - \frac{1}{2} T_5^5 \rangle = 0$ derived in ref. [21], and it has the same origin: the combination $e^{4\langle A \rangle} \langle T_0^0 + T_5^5 \rangle$ is constrained by Einstein equations to be a total derivative, therefore its integral over a compact interval should vanish. However, while $\int dx^5 e^{\langle A \rangle} \langle T_0^0 - \frac{1}{2} T_5^5 \rangle = 0$ also holds for a dS_4 or AdS_4 background, the condition $\int dx^5 e^{4\langle A \rangle} \langle T_0^0 + T_5^5 \rangle = 0$ is more specific for Poincaré invariance. To see how eq. (2.10) (with $\bar{\Lambda} = 0$) relates the vanishing of the 4d cosmological constant to the consistency of the 5d Einstein equations, let us note that in the Randall-Sundrum model [4] $r_c \int dx^5 e^{4\langle A \rangle} \langle T_0^0 + T_5^5 \rangle = (e^{-4\pi k r_c} - 1)\, \Lambda/k - e^{-4\pi k r_c} V_{obs} - V_{hid}$; therefore, the above constraint is automatically satisfied once the fine-tuning needed to achieve a Poincaré-invariant solution in four dimensions (namely $V_{hid} = -V_{obs} = \Lambda/k$) is imposed. A similar cancellation is expected to appear in the approach put forward in ref. [3, 22, 23].

3. Vacuum configurations with singularities and fine-tuning

In this section, we consider various vacuum configurations leading to singular warped compactifications, and show that in this case Einstein equations are not globally satisfied (condition (2.10) is not fulfilled), leading to an inconsistency of the solution. This implies in particular that the recently proposed "self-tuning" mechanism of the cosmological constant [9, 10] does not work as it stands, and hides a fine-tuning at the singularity.

[2] Note that for $\bar{\Lambda} = 0$ this is just the condition of vanishing vacuum energy imposed in [12].

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Let us first recall what is generally understood under "self-tuning of the cosmological constant" in brane-world models. It is the property that solutions of the five-dimensional Einstein equations preserving Poincaré invariance on the brane can be found for a wide range of values of the brane tension, i.e. independently of the brane contribution to the effective cosmological constant. This behaviour, first identified in ref. [9] and [10], has been shown to be generic in models where a single brane is coupled to gravity and a scalar field [16]. Indeed, plugging the ansatz

$$ds^2 = e^{2A(x^5)}\eta_{\mu\nu}\, dx^\mu dx^\nu + \left(dx^5\right)^2 \tag{3.1}$$

into the equations of motion (2.3) and (2.5) (which are now written for a single brane located at $x^5 = 0$), one generally finds solutions without imposing any relation among the parameters of the lagrangian. In particular, the brane tension $f(\phi)$ — which contains the contribution of the fields living on the brane (the SM fields) to the four-dimensional cosmological constant — needs not be fine-tuned and may vary over a wide range of values.

However, these solutions either do not localize gravity on the brane (the graviton zero mode is not a normalizable bound state, or equivalently the four-dimensional Planck mass diverges) or they have curvature singularities at finite proper distance.[3] At these singularities, the warp factor vanishes — implying that the four-dimensional metric degenerates, $G_{\mu\nu} \to 0$ — and the curvature scalar diverges, signaling a breakdown of Einstein gravity. In the absence of any mechanism that would smooth these singularities, one assumes that their effect is simply to cut off the fifth dimension, effectively compactifying it to a finite interval and ensuring localization of gravity on the brane. However, due to the presence of singularities the consistency condition (2.10) is not satisfied, which indicates that the ansatz (3.1) does not solve the equations of motion on the whole interval. This can be seen in explicit examples by the fact that the effective four-dimensional cosmological constant, eq. (2.9),

$$\lambda\, M_{\rm Pl}^2 = -\frac{2}{3}\int dx^5\, e^{4A}\, V(\langle\phi\rangle) - \frac{1}{3}\, e^{4A} f(\langle\phi\rangle)\,|_{x^5=0} \tag{3.2}$$

does not vanish, in spite of the Poincaré invariance of the geometry on the brane. Thus, in order for the solution to make sense, the (unknown) microphysics that resolves the singularity has to contribute to λ in such a way that it exactly cancels the brane and bulk contributions. This non-trivial statement ruins the apparent self-tuning behaviour of the solution: the traditional fine-tuning problem of the cosmological constant has simply been shifted into an unknown sector of the model. Of

[3] The authors of ref. [16] have shown that in this class of models, the only solutions that localize gravity with an infinitely large extra dimension involve a fine-tuning between bulk and brane parameters, like in the Randall-Sundrum model.

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course one cannot exclude that an adjustement mechanism, possibly involving some new light degrees of freedom [20, 16], may restore the self-tuning; but a concrete realization of this idea is still missing.

In order to make the problem explicit, one can parametrize the contribution of the unknown physics at the singularity by adding delta source terms in the energy-momentum tensor, $T_{MN} = -\sum_s f_s(\phi)\delta(x^5 - x_s) g_{\mu\nu}\delta^\mu_M \delta^\nu_N + \cdots$, where x_s denotes the position of a singularity. The necessity of adjusting precisely these terms for each particular solution (for each value of the brane tension) spoils the self-tuning. Indeed, in addition to the boundary conditions at the brane, the solution has to satisfy, at each singularity x_s, the following two constraints:

$$e^{4A(x_s)}\Big[\phi'(x_s+0) - \phi'(x_s-0)\Big] = \frac{3}{8}\, e^{4A(x_s)}\,\frac{\partial f_s}{\partial \phi}[\phi(x_s)]\,, \tag{3.3}$$

$$e^{6A(x_s)}\Big[A'(x_s+0) - A'(x_s-0)\Big] = -\frac{1}{6}\, e^{6A(x_s)} f_s[\phi(x_s)]\,. \tag{3.4}$$

The fine-tuning implied by eq. (3.3) and (3.4) guarantees that the consistency condition (2.10) is fulfilled and, as a consequence, that the 4d cosmological constant vanishes.

In ref. [12], this point was illustrated in the case of a vanishing bulk scalar potential, $V(\phi) = 0$, addressed in ref. [9] and [10]. In this section, we would like to discuss another example and to show that the requirement of having a global solution of the five-dimensional equations of motion in the presence of curvature singularities imposes a fine-tuning for more general ansätze than the (four-dimensional) Poincaré-invariant background (3.1).

Before doing so, a comment on the procedure of cutting off the 5th direction at the singularity is in order. The space has a boundary and new terms might have to be added to the action [24]. In the following, we circumvent this problem by keeping $-\infty < x^5 < \infty$ while continuing the solution beyond the singularities via the prescription $A(x^5) \equiv -\infty$ if x^5 is not between the two singularities. In computing the vacuum energy (or the effective cosmological constant) one integrates from $-\infty$ to ∞. Since the integrand vanishes when x^5 is not in between the two singularities, the space is effectively cut off there, while formally non compact. Alternatively, one may continue the solutions periodically and integrate over one period. Then the 5th direction has S^1 topology and hence no boundary (in order to continue periodically solutions that are not symmetric under $x^5 \leftrightarrow -x^5$ one first needs to reflect them at the singularities).

Let us first consider the case of a flat brane located at $x^5 = 0$, i.e. we take the Poincaré-invariant ansatz (3.1). In order to illustrate our point, we choose examples of bulk scalar potentials for which one can obtain a simple analytical expression for the solution of the equations of motion. The simplest case corresponds to a vanishing bulk potential and has been addressed in ref. [12]. Let us repeat shortly the discussion here. The solution of the bulk equations reads [10]

$$\phi\left(x^5\right) = \begin{cases} \frac{3}{4}\,\epsilon_1 \log\left|\frac{4}{3}x^5 + c_1\right| + d_1\,, & x^5 < 0 \\ \frac{3}{4}\,\epsilon_2 \log\left|\frac{4}{3}x^5 + c_2\right| + d_2\,, & x^5 > 0\,, \end{cases} \tag{3.5}$$

$$A'\left(x^5\right) = \begin{cases} \frac{1}{3}\,\epsilon_1\,\phi'\left(x^5\right), & x^5 < 0 \\ \frac{1}{3}\,\epsilon_2\,\phi'\left(x^5\right), & x^5 > 0\,, \end{cases} \tag{3.6}$$

where $\epsilon_{1,2} = \pm 1$ and c_1, c_2, d_1 and d_2 are integration constants. The boundary conditions on the brane determine three of them in terms of the fourth one and of the parameters in the lagrangian; this means in particular that there are solutions for a wide range of values of the brane tension and of the scalar coupling to the brane. The x^5-dependence of the warp factor ($e^{2A(x^5)} \propto |\frac{4}{3}x^5 + c_i|^{1/2}$) does not allow for a localization of gravity on the brane with an infinite extra dimension; therefore, the only phenomenologically acceptable solutions are the ones that have singularities on both axes $x^5 < 0$ and $x^5 > 0$, effectively compactifying the fifth dimension to a finite interval. In practice this means that we must choose $c_1 > 0$ and $c_2 < 0$; the singularities are then located at $x_- = -\frac{3}{4}c_1$ and $x_+ = -\frac{3}{4}c_2$. Following ref. [12], we continue the solution beyond the singularities by setting $|x^5 - x_-| \equiv 0$ for $x^5 < x_-$ and $|x^5 - x_+| \equiv 0$ for $x^5 > x_+$. As explained above, (3.5) and (3.6) do not solve the equations of motion on the closed interval $[x_-, x_+]$. This is reflected in the fact that, as can be immediately seen from eq. (3.2), the effective four-dimensional cosmological constant does not vanish, unless the brane tension itself vanishes. A simple way to cure this inconsistency is to "resolve" the singularities by adding the following source terms to the action (2.2):

$$-\int d^4x\,\sqrt{-g}\,f_-(\phi)\,\big|_{x^5=x_-} - \int d^4x\,\sqrt{-g}\,f_+(\phi)\,\big|_{x^5=x_+}\,. \tag{3.7}$$

The matching conditions (3.3) and (3.4) then amounts to a fine-tuning of the source terms at the singularities. Assuming for example exponential couplings for ϕ as in [9, 10], $f(\phi) = e^{b\phi}\,T$, $f_-(\phi) = e^{b_-\phi}\,T_-$ and $f_+(\phi) = e^{b_+\phi}\,T_+$, one obtains

$$b_- = \mp\frac{4}{3}\,, \qquad T_- = -2\,e^{\pm\frac{4}{3}d_1}\,, \qquad b_+ = \pm\frac{4}{3}\,, \qquad T_+ = -2\,e^{\mp\frac{4}{3}d_2}\,, \tag{3.8}$$

and $\epsilon_1 = -\epsilon_2 = \pm 1$ for $|b| < 4/3$ (solution (I) of ref. [10]), and

$$b_- = b_+ = \mp\frac{4}{3}\,, \qquad T_- = T_+ = -\frac{T}{2}\,, \tag{3.9}$$

and $\epsilon_1 = \epsilon_2 = \pm 1$ for $b = \mp 4/3$ (case considered in ref. [9] and solution (II) of ref. [10]). Note that since in the case $|b| = 4/3$ the solution is symmetric under $x^5 \leftrightarrow -x^5$, it is possible to identify the two singularities and to treat x^5 as a periodic coordinate (this amounts to continue periodically the solution beyond the singularities instead of "truncating" it as we did above). Then one needs to add a single energy source at $x_s \equiv x_- = x_+$, $-\int d^4x\,\sqrt{-g}\,e^{b_s\phi}T_s|_{x^5=x_s}$ with $b_s = \mp 4/3$ and $T_s = -T$.

The second explicit example [25] we would like to discuss corresponds to a constant bulk scalar potential, i.e. a non-vanishing bulk cosmological constant, $V(\phi) = \Lambda_B$. For the sake of simplicity, we focus on the case $\Lambda_B < 0$ and consider only $\mathbb{Z}_2$-symmetric solutions; when convenient we shall work on the half line $x^5 \geq 0$. The solution of the bulk equations reads [25]

$$e^{4A(x^5)} = e^{4A(0)} \left[\cosh\left(\omega|x^5|\right) + r \sinh\left(\omega|x^5|\right)\right], \tag{3.10}$$

$$\phi'(x^5) = c\, e^{-4[A(x^5)-A(0)]}, \qquad \phi(x^5) = c \int_0^{x^5} dy\, e^{-4[A(y)-A(0)]} + d\,, \tag{3.11}$$

$$c^2 = \frac{9}{16}\,\omega^2 \left[r^2 - 1\right], \tag{3.12}$$

where $\omega \equiv \sqrt{-\frac{4}{3}\Lambda_B}$. Note that the last equation requires $|r| \geq 1$. The boundary conditions on the brane give the following two relations:

$$f[\phi(0)] = -3\,\omega\, r\,, \qquad \frac{\partial f}{\partial \phi}\,[\phi(0)] = \frac{16}{3}\, c\,. \tag{3.13}$$

Plugging (3.13) into (3.12), one obtains a relation between $f[\phi(0)]$, $\frac{\partial f}{\partial \phi}[\phi(0)]$ and Λ_B. Given $f(\phi)$ and Λ_B, it is generally possible to find a value of $d \equiv \phi(0)$ which satisfies this constraint for a wide range of values of the brane tension $f[\phi(0)]$; eq. (3.13) then determines the other two integration constants r and c. For instance, if ϕ has only exponential couplings, $f(\phi) = e^{b\phi}\, T$, $\phi(0)$ is determined to be $\phi(0) = \frac{1}{2b} \ln[16\omega^2/T^2(16/9 - b^2)]$ (note the restriction on the parameter b, $|b| < 4/3$). Therefore, this solution has a self-tuning behaviour; however, it localizes gravity on the brane only if $r \leq -1$. In the case $r < -1$, there is a singularity at finite proper distance, located at

$$x_s = \frac{1}{\omega}\, \mathrm{arctanh}\left(-\frac{1}{r}\right) \tag{3.14}$$

(together with its twin singularity at $x^5 = -x_s$), and the solution can be rewritten

$$e^{4A(x^5)} = e^{4A(0)}\, \frac{\sinh\left[\omega(x_s - |x^5|)\right]}{\sinh(\omega x_s)}\,, \tag{3.15}$$

$$\phi(x^5) = -\frac{3}{4}\,\epsilon\, \log\left(\frac{\tanh\left[\,\frac{\omega}{2}(x_s - |x^5|)\,\right]}{\tanh\left(\frac{\omega x_s}{2}\right)}\right) + d\,, \tag{3.16}$$

where $\epsilon = \mathrm{sign}(\frac{\partial f}{\partial \phi}[\phi(0)])$. In the case $r = -1$, the solution becomes $e^{4A(x^5)} = e^{4A(0)} e^{-\omega|x^5|}$ and the singularity is pushed to infinity. This limit corresponds to a constant ϕ background (i.e. $\phi(x^5) = \mathrm{cst}$, as implied by eq. (3.12)), and eq. (3.13) amounts to a fine-tuning between the bulk cosmological constant Λ_B and the brane tension $V_0 \equiv f[\phi(0)]$, namely $V_0 = 3\,\omega$. It is interesting to note [25] that this fine-tuning is precisely the one appearing in the Randall-Sundrum model (here the configuration would be the one considered in ref. [5], in which the second brane is pushed to infinity; note that our notations correspond to setting $2M = 1$ in the formula of Randall and Sundrum). This difference of behaviour of the two solutions (fine-tuning versus self-tuning) is already striking, given the fact that the fine-tuned solution can be obtained from the "self-tuning" solution by taking the limit $x_s \to \infty$. If one now computes the effective four-dimensional cosmological constant, one finds, using (3.2),

$$\lambda\, M_{\rm Pl}^2 = -\,\frac{e^{4A(0)}\,\omega}{\sinh(\omega x_s)}\,. \tag{3.17}$$

Thus, in the case $r < -1$ ("self-tuning" solution with a singularity at finite proper distance), there is only a partial cancellation between the bulk and the brane contributions to the 4d cosmological constant, while in the case $r = -1$ (fine-tuned solution with a singularity at infinity) $\sinh(\omega x_s) = \infty$ and therefore $\lambda = 0$. This resolves the apparent paradox noted above: the "self-tuning" solution is incomplete, i.e. an additional source has to be added at the singularity in order for the equations of motion to be globally satisfied. Specifically, adding $-\int d^4x\,\sqrt{-g}\,e^{b_s\phi}T_s|_{x^5=x_s}$ to the action, one finds that a fine-tuning of b_s and T_s is required,

$$b_s = \frac{4}{3}\,\epsilon\,, \qquad T_s = -\frac{3}{2}\,\frac{\omega\, e^{-b_s d}}{\tanh\left(\frac{\omega x_s}{2}\right)}\,. \tag{3.18}$$

The contribution of the singularity to the 4d cosmological constant then exactly cancels (3.17), as expected. In the limit $x_s \to \infty$, this contribution vanishes.

Another way to understand the necessity of adding an energy source at the singularity is to consider a slightly different set-up, in which a second "end of the world" brane is placed at a position $x^5 = L < x_s$. Then eq. (3.13) must be supplemented with a set of new boundary conditions at $x^5 = L$,

$$f_L[\phi(L)] = -3\,\omega\,\coth\left[\omega(x_s - L)\right], \tag{3.19}$$

$$\frac{\partial f_L}{\partial\phi}\,[\phi(L)] = -\,\frac{16}{3}\,c\,\frac{\sinh(\omega x_s)}{\sinh\left[\omega(x_s - L)\right]}\,. \tag{3.20}$$

Since the integration constants d, r (hence x_s) and c are determined by the boundary conditions at $x^5 = 0$, eq. (3.19) and (3.20) imply a fine-tuning of $f_L(\phi)$, which at the same time fixes the inter-brane distance L, in the same way as in ref. [26]. More specifically, for an exponential coupling of the bulk scalar field $f_L(\phi) = e^{b_L\phi}\,T_L$, eq. (3.19) and (3.20) yield

$$b_L = \frac{4}{3}\,\epsilon\,\cosh^{-1}\left[\omega(x_s - L)\right], \tag{3.21}$$

$$T_L = -3\,\omega\,e^{-b_L d}\coth\left[\omega(x_s - L)\right]\left(\frac{\tanh\left[\frac{\omega}{2}(x_s - L)\right]}{\tanh\left(\frac{\omega x_s}{2}\right)}\right)^{\cosh^{-1}[\omega(x_s - L)]}\,. \tag{3.22}$$

Thus, for a given value of b_L (resp. T_L), eq. (3.21) (resp. eq. (3.22)) determines the inter-brane distance L, while eq. (3.22) (resp. eq. (3.21)) tells us that the brane vacuum energy T_L (resp. the scalar coupling b_L) must be fine-tuned. Conversely, for a fixed value of L, both b_L and T_L must be fine-tuned. If one now sends the second brane to the singularity, $L \to x_s$, one finds exactly $b_L \to b_s$ and $T_L \to T_s$: the "self-tuning" single-brane set-up is nothing but the (singular) limit of a fine-tuned two-brane system, in which the second brane hides at the singularity.[4]

Note that we could "screen" the singularities in the same way in the case of a vanishing scalar potential discussed previously, i.e. we could place two branes at $x^5 = L_- > x_-$ and at $x^5 = L_+ < x_+$. Unlike what we have just found for a constant scalar potential, we would then conclude that the brane tensions and scalar couplings do not depend on the positions of the two additional branes, i.e. eq. (3.8) (resp. eq. (3.9) in the $\mathbb{Z}_2$ symmetric case) hold for any inter-brane distances, and L_-, L_+ are moduli. The statement that one recovers the "self-tuning" solution in the limit $L_- \to x_-$, $L_+ \to x_+$ follows then trivially.

We could go on discussing other examples with a more general bulk scalar potential; however, while the explicit form of the solutions would become more involved, the conclusions would remain the same.

We would now like to show that the necessity of adding energy sources in order to ensure the consistency of a solution of Einstein equations with singularities is not a particularity of the ansatz (3.1). For this purpose, we come back to the case of a vanishing bulk scalar potential, in which curved solutions with maximal symmetry (de Sitter or Anti-de Sitter) in four dimensions can be found [20]. Our ansatz is

$$ds^2 = e^{2A(x^5)}\,\bar{g}_{\mu\nu}\,dx^\mu dx^\nu + \left(dx^5\right)^2\,, \tag{3.23}$$

where $\bar{g}_{\mu\nu}$ is the metric of a maximally symmetric $(3+1)$-dimensional space with a curvature constant $\bar{\Lambda}$ (the explicit form of $\bar{g}_{\mu\nu}$ has been given at the beginning of section 2). With this ansatz, the solution of the bulk equations of motion can be written in terms of a hypergeometric function $F(z) \equiv {}_2F_1(1/2, 2/3, 5/3, z)$ [20]

$$\begin{aligned} \frac{1}{\gamma_i}\,e^{4A(x^5)}\,F\left(-\frac{9\bar{\Lambda}}{\gamma_i^2}\,e^{6A(x^5)}\right) &= \epsilon_i\left(\frac{4}{3}\,x^5 + c_i\right)\,, \\ \phi'(x^5) &= \gamma_i\,e^{-4A(x^5)}\,, \end{aligned} \tag{3.24}$$

where $i = 1, 2$ refer to $x^5 < 0$ and $x^5 > 0$ respectively, $\epsilon_{1,2} = \pm 1$, c_1, c_2 are integration constants and $\gamma_i = \epsilon_i F|_{x^5=0}/c_i$ if we choose $A(0) = 0$. The boundary conditions on

[4]It is interesting to consider another limit, in which the singularity is pushed to infinity, while the second brane remains at a finite proper distance L from the brane at the origin. Then one recovers again the Randall-Sundrum model (in the configuration of ref. [4]), with the double fine-tuning $V_0 = -V_L = 3\,\omega$, where $V_L \equiv \lim_{x_s \to \infty} f_L[\phi(L)]$.

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the brane give the following two constraints:

$$f[\phi(0)] = 2\left(\sqrt{\gamma_1^2 + 9\bar\Lambda} + \sqrt{\gamma_2^2 + 9\bar\Lambda}\right), \qquad \frac{\partial f}{\partial \phi}[\phi(0)] = \frac{8}{3}(\gamma_2 - \gamma_1). \tag{3.25}$$

Thus, given $f(\phi)$, one generally finds a continuous set of solutions corresponding to different values of $\bar\Lambda$ (including a flat solution $\bar\Lambda = 0$). The authors of ref. [9] and [20] have shown that for an exponential coupling of the scalar field to the brane, there is a particular value of the coupling which does not allow for any curved solution. Indeed, pluging $f(\phi) = e^{b\phi}T$ in the above constraints (3.25), one sees that only $\bar\Lambda = 0$ leads to a solution in the case $b = \pm 4/3$, while any $\bar\Lambda$ is allowed for $|b| < 4/3$ (and any $\bar\Lambda < 0$ is allowed for $|b| > 4/3$). Here we are interested in curved solutions, so we choose $b \neq \pm 4/3$ and consider $\bar\Lambda \neq 0$. The solutions have singularities on both axes if we take $c_1 > 0$ and $c_2 < 0$, like in the $\bar\Lambda = 0$ case. The vacuum energy is given by eq. (3.2) with an additional term $12\bar\Lambda M_{\rm Pl}^2$ on the r.h.s.; assuming first that the singularities do not contribute, one finds

$$\lambda M_{\rm Pl}^2 = -\frac{2}{3}\left(\sqrt{\gamma_1^2 + 9\bar\Lambda} + \sqrt{\gamma_2^2 + 9\bar\Lambda}\right) + 12\bar\Lambda M_{\rm Pl}^2, \tag{3.26}$$

while the Planck mass is computed to be

$$M_{\rm Pl}^2 = \frac{1}{9\bar\Lambda}\left[\left(\sqrt{\gamma_1^2 + 9\bar\Lambda} + \sqrt{\gamma_2^2 + 9\bar\Lambda}\right) - (|\gamma_1| + |\gamma_2|)\right]. \tag{3.27}$$

Since $\bar R = 12\bar\Lambda$, the four-dimensional Einstein equations require $\lambda = 6\bar\Lambda$. Clearly this is not the case here, which indicates an inconsistency of the solution. Adding now delta energy sources $f_\pm(\phi)$ at the singularities, one obtains two new sets of two boundary conditions,

$$e^{4A} f_\pm(\phi)\Big|_{x^5 = x_\pm} = -2\,|\gamma_{|^2_1}|\,, \tag{3.28}$$

$$e^{4A}\frac{\partial f_\pm}{\partial \phi}(\phi)\Big|_{x^5 = x_\pm} = \mp\frac{8}{3}\,\gamma_{|^2_1}\,, \tag{3.29}$$

where $\gamma_{|^2_1} = \gamma_1$ for $x^5 = x_-$ and γ_2 for $x^5 = x_+$. It is now straightforward, using eq. (3.28), to check that the singularities give a contribution $\frac{2}{3}(|\gamma_1| + |\gamma_2|)$ to $\lambda M_{\rm Pl}^2$, therefore ensuring $\lambda = 6\bar\Lambda$, as required by the 4d Einstein equations. Note that, like in the $\bar\Lambda = 0$ case, the boundary conditions at the singularities introduce a fine-tuning which is naively absent if one does not insist on the validity of Einstein equations at the singularities. Thus even curved solutions require a fine-tuning.

4. More on nearby curved solutions

In the previous section we have seen that solutions whith singularities are not consistent unless one specifies the physics at the singularity. Our proposal was to interprete the corresponding curvature singularities as arising due to the presence of additional source terms. This proposal passed the consistency condition (2.9). The lesson we learned is that fine tuning on the parameters of sources at the singularities is needed. Therefore, these setups do not solve the cosmological constant problem. However, irrespective of the fine-tuning problem, they possess an additional property which makes them attractive in view of the cosmological constant problem: for the very specific choice of parameters (namely vanishing bulk potential and particular value of the scalar-brane coupling) discussed in [9] and in solution (II) of [10], there are no nearby curved solutions [9, 20]. In the present section we are going to rederive that result in a simpler way, and afterwards we will explore what happens when one adds a bulk potential for the scalar field ϕ.

The action considered in [9, 10] reads

$$S = \int d^5x\sqrt{-G}\left[R - \frac{4}{3}\left(\nabla\phi\right)^2\right] - \int_{x^5=0} d^4x\sqrt{-g}Te^{b\phi}\,. \tag{4.1}$$

In our previous notations (2.2), this corresponds to $V(\phi) = 0$ and $f(\phi) = Te^{b\phi}$. This is some version of five-dimensional Jordan-Brans-Dicke (JBD) theory in the Einstein frame where the scalar ϕ couples in a universal manner (as a conformal factor of the metric) to matter. However, all matter is confined to live on a 4d hypersurface located at $x^5 = 0$. The metric on this hypersurface is taken as the induced one. We define the JBD frame by performing a Weyl transformation such that ϕ decouples from all matter. This is achieved by the field redefinition $G_{MN} \to e^{-b/2\phi}G_{MN}$. In the JBD frame the action reads,

$$S = \int d^5x\sqrt{-G}e^{-\frac{3}{4}b\phi}\left[R + \left(\frac{3}{4}b^2 - \frac{4}{3}\right)\left(\nabla\phi\right)^2\right] - \int_{x^5=0} d^4x\sqrt{-g}T\,. \tag{4.2}$$

Now, for $b = \pm 4/3$ we observe that the ϕ equation of motion results in the constraint $R = 0$ and we see that this case is special. It corresponds exactly to the setup where only flat solutions exist [9, 20]. Now, we will re-derive this result in the JBD frame (4.2) where the calculation turns out to simplify substantially. This will allow us to study how the solution is affected by the presence of a bulk scalar potential. We should remark that in this section we understand that, in the spirit of the previous section, sources at singularities need to be added. However, in order to keep formulas short we do not write down those terms explicitly. Typically we are looking for solutions of the form ($\mu, \nu = 1, \ldots, 4$)

$$ds^2 = e^{2A\left(x^5\right)}\bar{g}_{\mu\nu}\left(x^\mu\right)dx^\mu dx^\nu + \left(dx^5\right)^2\,, \tag{4.3}$$

and

$$\phi = \phi\left(x^5\right) \tag{4.4}$$

(a non-trivial 55 component of the metric which may appear after the Weyl transformation can be absorbed in a coordinate transformation). Further, we chose $\bar{g}_{\mu\nu}$

to be the metric of a maximally symmetric $(3+1)$-dimensional space

$$\bar{R}_{\mu\nu\kappa\lambda} = \bar{\Lambda}\Big(\bar{g}_{\mu\kappa}\bar{g}_{\nu\lambda} - \bar{g}_{\mu\lambda}\bar{g}_{\nu\kappa}\Big)\,, \tag{4.5}$$

where the bar indicates that the Riemann tensor is computed with respect to the metric $\bar{g}_{\mu\nu}$, whose explicit form was given in section 2.

The equations of motion derived from (4.2) for $|b| = 4/3$ are (as in the previous sections, a prime denotes derivative with respect to x^5)

$$-3\bar{\Lambda}e^{-2A} + 2A'' + 5A'^2 = 0\,, \tag{4.6}$$

$$-6\bar{\Lambda}e^{-2A} + 6A'^2 - \frac{4b}{|b|}A'\phi' = 0\,, \tag{4.7}$$

$$-3\bar{\Lambda}e^{-2A} + 3A'' + 6A'^2 - 3\frac{b}{|b|}A'\phi' - \frac{b}{|b|}\phi'' + \phi'^2 + \frac{1}{2}Te^{\frac{b}{|b|}\phi}\delta\left(x^5\right) = 0\,, \tag{4.8}$$

where (4.6) is the ϕ equation of motion, and (4.7), (4.8) are the 55, and $\mu\nu$ components of the Einstein equations, respectively. We are looking for solutions with continuous functions A and ϕ. The advantage of the JBD frame is that the source term appears only in one of eqs, (4.8). From (4.6) we see immediately that A' must be continuous such that A'' is finite. With (4.7) follows that either ϕ' is also continuous or $A'(0) = 0$. If ϕ' is continuous then (4.8) implies that T has to vanish — a case in which we are not interested. Therefore we take $A'(0) = 0$ which restricts the value of $\bar{\Lambda}$ to vanish. So, without solving explicitly any differential equation we rederived that for $|b| = 4/3$ there are no nearby curved solutions. Just for completeness we give the solution to the remaining equation (4.8). Solving in the bulk gives

$$A(x^5) = \text{const}\,, \qquad \phi = -\frac{b}{|b|}\log\left|\alpha_i x^5 + \beta_i\right|\,, \tag{4.9}$$

where the index $i = 1, 2$ labels the case of $x^5 < 0$, $x^5 > 0$, respectively. Using the $\mathbb{Z}_2$ symmetry of the modulus function we can without loss of generality restrict to the case that $\alpha_i > 0$. Further, we want to have a finite 4d Planck mass, and this forces us to consider a solution with a singularity appearing for some value of x^5. Then continuity at zero leads to,

$$\beta_1 = -\beta_2 = \beta > 0\,. \tag{4.10}$$

The jump condition on the first derivative of ϕ at zero finally restricts

$$\alpha_1 + \alpha_2 = \frac{1}{2}T \tag{4.11}$$

(for a symmetric solution this implies $\alpha_1 = \alpha_2 = \frac{1}{4}T$). Transforming back to the Einstein frame and performing an appropriate coordinate transformation in the fifth direction one finds easily that this is solution II of [10], or equivalently the solution discussed in [9] (i.e. one recovers eq. (3.5) and (3.6) with $\epsilon_1 = \epsilon_2 = -b/|b|$, $c_1 = -c_2$ and $d_1 = d_2$).

As a next step we want to investigate how stable the constraint $\bar{\Lambda} = 0$ is against deformations of the bulk lagrangian. To this end, we will focus on the special case of adding an exponential bulk potential for the dilaton ϕ, insisting, however, on $|b| = 4/3$. (To some extent this mimics also the option of screening the singularities by additional branes — the additional bulk potential can be viewed as filling the space with a continuum of three-branes.) To be specific, we modify the action (4.2) as follows:

$$S = \int d^5x\sqrt{-G}\left[e^{-\frac{3}{4}b\phi}R - \Lambda e^{(a-\frac{5}{4}b)\phi}\right] - \int_{x^5=0} d^4x\sqrt{-g}T\,, \tag{4.12}$$

which corresponds to adding a bulk potential $V(\phi) = \Lambda e^{a\phi}$ in the original Einstein frame. The equations of motion with the ansatz (4.3), (4.4) read

$$-3\bar{\Lambda}e^{-2A} + 2A'' + 5A'^2 + \frac{5b-4a}{12b}\Lambda e^{(a-b/2)\phi} = 0\,, \tag{4.13}$$

$$-6\bar{\Lambda}e^{-2A} + 6A'^2 - \frac{4b}{|b|}A'\phi' + \frac{1}{2}\Lambda e^{(a-b/2)\phi} = 0\,, \tag{4.14}$$

$$\begin{aligned} -3\bar{\Lambda}e^{-2A} + 3A'' + 6A'^2 - 3\frac{b}{|b|}A'\phi' - \frac{b}{|b|}\phi'' + \\ + \phi'^2 + \frac{1}{2}\Lambda e^{(a-b/2)\phi} + \frac{1}{2}Te^{\frac{b}{|b|}\phi}\delta\left(x^5\right) = 0\,. \end{aligned} \tag{4.15}$$

Together with the requirement of having continuous functions A and ϕ the first equation (4.13) implies that A' must be continuous as well. In order to allow for a non-vanishing T in the third equation (4.15) we need a jump in ϕ' at $x^5 = 0$. From the second equation (4.14) one sees that this is possible only for

$$A'(0) = 0\,, \tag{4.16}$$

and that any solution has to satisfy the condition

$$6\bar{\Lambda}e^{-2A(0)} = \frac{1}{2}\Lambda e^{(a-b/2)\phi(0)}\,. \tag{4.17}$$

This is quite a remarkable result. Insisting on $|b| = 4/3$ but allowing for non-zero Λ we find that the effective 4d cosmological constant $\bar{\Lambda}$ necessarily differs from zero. The situation is somewhat complementary to the case $|b| < 4/3$. There nearby curved solutions exist even for vanishing Λ [20]. On the other hand it is possible to have flat solutions for non-vanishing Λ (solution III in [10] and generalizations thereof [16]). To complete the discussion, we now solve the equations in the particular case where the warp factor is constant (which trivially satisfies the constraint $A'(0) = 0$), and show that the corresponding solution gives the flat solution (4.9) in the limit $\Lambda \to 0$. Plugging $A'(x^5) = 0$ into eqs. (4.13) and (4.14), one finds that they can be solved only if

$$a = \frac{b}{2} \tag{4.18}$$

(conversely, if we choose the bulk scalar potential to satisfy (4.18), the equations of motion imply that $A'(x^5) = 0$). Condition (4.17) then becomes

$$\bar{\Lambda} e^{-2A(0)} = \frac{1}{12}\Lambda \,. \tag{4.19}$$

For $\Lambda < 0$ we find

$$\phi = \begin{cases} -\frac{b}{|b|} \log \left| \frac{\alpha_1 \sinh \rho x^5}{\rho} + \beta \cosh \rho x^5 \right| , & x^5 < 0 \\ -\frac{b}{|b|} \log \left| \frac{\alpha_2 \sinh \rho x^5}{\rho} - \beta \cosh \rho x^5 \right| , & x^5 > 0 \,, \end{cases} \tag{4.20}$$

where $\rho = \sqrt{-\Lambda/4}$. In order to have singularities we chose $\alpha_i, \beta > 0$. The function is continuous, and the jump conditions on the first derivatives leads to (4.11). For $\Lambda > 0$ we replace in (4.20) the hyperbolic functions by their trigonometric counterparts and $\rho = \sqrt{\Lambda/4}$. The limit $\Lambda \to 0$ corresponds to $\rho \to 0$, and one easily rediscovers (4.9).

5. Further constraints on self-tuning models

Up to now, we have discussed the self-tuning proposal at the level of an effective classical lagrangian. We did not specify the relation of these classical models to a realistic quantum field theory. In particular, making such a relation one needs to specify the meaning of the brane tension in terms of the perturbative lagrangian for matter fields, and any other, non perturbative in 5d, pieces of the lagrangian on the brane. Firstly, as we argue later on, there may be some primordial non-zero contribution to the tension on any wall, which is the remnant of the higher-dimensional theory, and can be considered as non dynamical and non perturbative in five dimensions. Such contributions do not even need to be a part of a supersymmetric sigma model on the brane, so they are not protected, or set to zero, by brane supersymmetry. Further, there is the matter lagrangian which has perturbative couplings and which is modified by quantum corrections on the brane. This lagrangian may be globally supersymmetric, or may just be the one of the usual Standard Model. There are basically two valid points of view on the place of the quantum corrections in the effective classical brane tension. One approach is based on the observation that there exist (fine-tuned) Poincaré invariant solutions for a wide range of couplings of the bulk scalar to the wall. Then, even if each loop contribution on the brane comes with a different dependence on ϕ, the expectation value of the sum of them up to any arbitrarily choosen order of the perturbation theory can be identified with brane tension in our models and efficiently screened. This point of view is valid in the present paper up to the section 4, and we take in that part the brane tension to be $-e^{b\phi}\int_{M_{(4)}}(\langle V_{\rm eff}^{(0)\cdots(n)}\rangle + T_p)$, where T_p is the primordial brane tension. However, in such a situation the Poincaré invariant solution is not the only one, there exist nearby curved solutions which give non-vanishing cosmological constant in 4d. In a more ambitious approach, one would demand, that there should be no nearby curved solutions. This, as shown in [9], requires a very careful choice of the the brane model and its coupling to the bulk scalar. There one has to assume that the classical brane tension is just the sum of some primordial tension and the vacuum value of the classical lagrangian for fields living on the brane, and watch explicitly the quantum corrections to see whether the required form of the model does not get disturbed. To be more specific, we saw in the previous section that if we want to achieve certain uniqueness of Poincaré invariant solutions, it is important to have a very particular value of b, like $|b| = 4/3$ of the previous section. It has been argued in [9] that it is possible to write down a model where quantum corrections respect the value of b once it is set at tree-level. The associated stability of the Poincaré invariant solution with respect to curved deformations is in fact the second part of the original self-tuning proposal. Hence it is instructive to take a closer look at the specific brane model assumed in [9]. The action integral is $\int d^5x\sqrt{-G}(R - \frac{4}{3}(\partial\phi)^2) + \int d^5x\sqrt{-\bar{g}}\delta(x^5)L_b(H;\bar{g}_{\mu\nu})$ where $\bar{g}_{\mu\nu} = g_{\mu\nu}e^{\frac{b}{2}\phi}$, $g_{\mu\nu}$ is the induced metric on the brane and $|b| = 4/3$. The special feature of this model is the coupling of the scalar field to the brane in its equation of motion

$$\frac{8}{3}(\sqrt{-G}\phi')' - \frac{b}{4}\sqrt{-g}\delta(x^5)\theta^\mu_\mu = 0\,, \tag{5.1}$$

where θ^μ_μ is the trace of the brane energy momentum tensor, defined by $\delta(\int d^4x\sqrt{-g}L_b$ $(H;g_{\mu\nu})) = -\frac{1}{2}\int d^4x\sqrt{-g}\,\theta_{\mu\nu}\delta g^{\mu\nu}$. This includes contributions from the dynamical sector containing the SM fields, and that from any primordial brane tension T_p, $(\theta^\mu_\mu)_p = 4T_p e^{b\phi_0}$. Let us note that if we were in four dimensions, the equation of motion for the scalar which couples to the trace of the energy-momentum tensor only would automatically imply vanishing of that trace on-shell, hence the vanishing of the cosmological constant. However, in the present case the vacuum value of derivatives of the scalar with respect to the coordinate transverse to the brane can be non zero without violating the Poincaré invariance. Hence, the equation of motion does not demand the vanishing of the trace of the brane energy-momentum tensor. In fact this is the situation we consider in this paper. However, the non vanishing of θ^μ_μ implies the scale anomaly on the brane which in turn implies that the Weyl rescaling on the brane is anomalous. This means that the conjectured decoupling of the scalar from the brane, hence "conservation" of b by brane physics, may be violated in the full quantum brane theory. In this respect, we agree with the argumentation put forward in ref. [14]. On the other hand, one could imagine choosing the brane sector which is conformally invariant to all orders, like $N = 4$ Super-Yang-Mills models. This implies $\theta^\mu_\mu = 0$ at all orders and the background cosmological constant vanishes as a result of a symmetry. In a realistic model however, such a symmetry has to be broken, and the cosmological constant problem reappears.

It is interesting to write down explicitly the 1-loop corrections to the effective potential in an example of a model conformally coupled to the scalar. To this end we recall that the ultimate contribution to the vacuum energy which must be cancelled in order to be consistent with observation, arises in low-energy theory, say around the 1 TeV scale. At that scale in all realistic models 4d brane symmetries which may have something to do with protecting the vacuum energy, like $N = 1$ supersymmetry or conformal invariance, are broken. This is the place where the mechanism of self-tuning (if not fine-tuned) would be really useful. To see what happens when quantum effects are taken into account, let us specify the representative content of the brane to be given as

$$e^{b\phi}\int\sqrt{-g}\Big(e^{-b/2\phi}g^{\mu\nu}\partial_\mu\bar\psi\partial_\nu\psi - m^2\bar\psi\psi + \\ + ie^{-b/4\phi}e^\mu_a\bar\lambda\gamma^a\partial_\mu\lambda - M\bar\lambda\lambda - \frac{1}{4g^2}e^{-b\phi}F^2 - V_0 - T_p\Big). \qquad (5.2)$$

To compute one-loop contribution to the effective potential we first perform the redefinition of the fields so that their kinetic terms are canonical with respect to Minkowski background on the brane, and then use the standard formulae with a physical UV cut-off Λ which we understand to be close to 1 TeV. After going back to the original frame one obtains $-\int\sqrt{-g}(e^{b\phi}(V_0+T_p)+\mathrm{Str}\,\mathbf{1}\,\mathcal{O}(\Lambda^4)+\frac{1}{32\pi^2}e^{b/2\phi}\Lambda^2(m^2 - M^2)) + \cdots$, where one can see terms which are multiplied by different powers of $e^{b\phi}$. Of course, this situation is also manifest in models without conformal coupling of the brane to the scalar field. Hence, in general, even if one carefully adjusts $|b| = 4/3$, at tree level, one departs fom this choice after including quantum corrections. This simply means that the classical selection rules have limited value in the true, quantum, world.

We want to mention another troublesome feature of the model with conformal coupling of the scalar to the branes, which is an inconsistency with long range properties of 4d gravity at the classical level. This becomes clear in the Jordan-Brans-Dicke frame introduced in the section 4. In that frame the model of [9] takes the form of a generic Jordan-Brans-Dicke model, and is subject to constraints given by the analysis of 4d Jordan-Brans-Dicke systems performed in [29]. The 4d Jordan-Brans-Dicke action in the JBD frame looks as follows: $\int d^4x\sqrt{-g}(\frac{\Phi^2}{2}R - 2\omega(\Phi)g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - V(\Phi))$ with the assumption that matter fields are coupled only to the metric $g_{\mu\nu}$ and not to Φ. This is very similar to the lagrangian given in (4.2), up to a redefinition of the field ϕ and integration over the fifth dimension. It is easy to see, through the integration of the fifth dimension directly in the 5d JBD frame, that in the case of (4.2) $\omega = 0$. It has been shown in [29] that there exist constraints on the value of the parameter ω of JBD models. This constraint says that $\frac{1}{2\omega+3} < 10^{-3}$, hence in the model which we are discussing here the above bound is violated. The troublesome constraint could be avoided if the 4d JBD scalar would receive a mass of the order of $1/mm$. This can be achieved by introducing a very soft potential for the scalar field, so that the experimental bound on the 4d cosmological constant, $\rho_\Lambda < (0.002\,\mathrm{eV})^4$ is not violated. At this point however, we should recall the result of the preceding section, where we have shown that introducing a simple bulk potential while insisting on $|b| = 4/3$ has led one from a flat to a curved solution.

However, one should repeat, that in the context of generic models lacking conformal coupling of branes to a bulk scalar, it is still likely that even for a modified, so pretty generic, dependence on ϕ there exists a Poincaré invariant vacuum background. The problematic point is, that in response to quantum corrections the tensions on the additional branes located at singularities must get re-finetuned. This would have to be achieved by some mechanism operating at low energies in addition to the bulk scalar mediation. One may speculate that the bulk KK modes could be an agent of such an readjustment, but a separate calculation is needed to decide whether this is the case.

The next issue which needs to be addressed is the physics at or near branes located at singularities. One way to conceal this problem is to say that the singularities are resolved in the microscopic higher dimensional, $D > 5$, model. However, perhaps one can still make some useful observations looking more carefully at the 5d models. A way to test physics near singularities is to perturb the model by a small point-like test mass and to read off the strength of interaction of this test mass with the gravitational field. The action for such a test particle is $S_p = -m \int ds$ in 5d coordinates and it gives a contribution to the energy momentum tensor of the form $T^{MN} = \frac{m}{\sqrt{-g}} \int d\tau \frac{\dot{x}^M \dot{x}^N}{\sqrt{-g_{PQ}\dot{x}^P \dot{x}^Q}} \delta(x - x(\tau))$. The perturbation of the action under a small variation $\delta h_{\mu\nu}$ is $\delta S_p = -\int d^5x \sqrt{-g}(e^{2A}\delta h_{\mu\nu}) = -\int d^4x \sqrt{-\tilde{g}} \delta h_{\mu\nu} \tilde{T}^{\mu\nu}$ where $\tilde{T}^{\mu\nu} = \int dy e^{6A} T^{\mu\nu}$. One obtains then the 4d energy momentum tensor due to a test mass $\tilde{T}^{\mu\nu} = \frac{m e^{A(x^5)}}{\sqrt{-\tilde{g}}} \int d\tau \frac{\dot{x}^\mu \dot{x}^\nu}{\sqrt{-\tilde{g}_{PQ}\dot{x}^P \dot{x}^Q}} \delta(x - x(\tau))$. This means that the mass which interacts with the gravitational field at the location x^5 is $m e^{A(x^5)}$, hence, for instance, it vanishes whenever $e^A = 0$. Hence, physical matter located at singular branes is effectively massless. Essentially, since $\sqrt{-G} = 0$ at the singular brane, local dynamics there is also suppressed. Then the mechanism which creates correlations between vacuum tensions on physical and singular branes might be the result of some higher-dimensional consistency condition, rather than a dynamical one. For instance, one can recall the way the Bianchi identity is fullfilled in the Horava-Witten model. There one needs to switch on vacuum fluxes of the gauge fields along the internal dimensions. The fluxes are different on different walls, but their sum is fixed by the Bianchi identity. These fluxes, together with that of gravitational curvature tensor in six compact dimensions, lead to non zero but correlated brane tensions in five space-time dimensions.

To learn something about dynamics in singular warped universe, it is useful to compute in the non-relativistic limit the force acting on a test mass m. From the geodesic equation one easily obtains the acceleration of the freely falling particle $\frac{d^2X^M}{dt^2} = -\Gamma^M_{00} = -\frac{1}{2}g^{MN}\frac{\partial g_{00}}{\partial x^N}$. This is non vanishing along the fifth direction and gives $\frac{d^2X^5}{dt^2} = A'(x^5)e^{2A(x^5)}$. Hence, the force acting on the slowly moving test particle is $F^5 = mA'(x^5)e^{2A(x^5)}$. To see the possible implications let us take an example of the singular background which has been discussed in the third section of this paper, in the vicinity of the naked singularity on the positive side of the x^5 axis, $e^A = e^{1/3d}|4/3y - c|^{1/4}$. When the particle approaches the singularity from the left, it feels the force $F^5 \approx -|4/3y - c|^{-1/2}$ which repells the test particle away from the singularity. This may be interpreted as a hint at sort of stability of the system, in the sense that one does not expect the flow of physical matter from the positive-tension brane to the singular negative tension branes.

Finally, let us comment briefly on the stabilization of the fifth dimension in the static (flat) case. The standard way to formulate this problem is to separate a x^5-independent part of G_{55}, let us call it here $r_c(x)$, and to require that after integrating out all remaining degrees of freedom and integrating over x^5 there remains an effective potential for r_c such that either it has a minimum (and thus truly stabilizes r_c) or it is flat, so no effective force moves r_c around. This second possibility would correspond for instance to a modulus in a supersymmetric model, but we know that at low energies supersymmetry needs to be completly broken down. Hence, of actual interest is the first one. The examples of that favourable situation are the models (with repaired singularities) presented in this paper which contain a non-trivial bulk potential for the scalar field. In the case of the models with no potential in the bulk, the distance to the singularities was not fixed. Moreover, if one puts the brane screening the singularity between the SM brane and singularity, its position is also undetermined. According to what has been said earlier, we expect quantum corrections to modify the classical solution in these cases. Parenthetically, we would like to notice that in the 5d Einstein frame in which we work in the initial sections of the paper, the substitute of the equation of motion determining r_c is simply the Einstein equation $E_{55} - \frac{1}{2}T_{55} = 0$. In addition, for the general class of solutions discussed here, it has been shown in ref. [28] that they are stable in the sense that there are no tachyonic fluctuations.

The remaining constraint on singular warped universes which we want to mention arises when one computes the corrections to Newton's law due to the exchange of heavy gravitons [27]. When one tries to send the singularities far apart, i.e. when one considers a macroscopic fifth dimension, then the correction to the effective 4d

Newton's law is $\delta V = -\frac{1}{r^2}\mu G_N \frac{|c_1|+|c_2|}{2\pi}$. Hence, the correction grows linearly with the size of the fifth dimension, similarly to what would happen in a toroidal compactification. This means that the distance between singularities should be smaller than, say, 1 mm. In this regime the correction due to heavy gravitons is exponentially suppressed as $\delta V = -\frac{1}{2r}\mu G_N e^{-\frac{\pi r}{2(|c_1|+|c_2|)}}(1+\sin(\frac{\pi|c_1|}{|c_1|+|c_2|}))$.

6. Summary and conclusions

In the present paper we have argued that in brane worlds the cosmological constant problem is merely seen from a new perspective. The degree of fine-tuning needed to obtain a value in agreement with observational bounds is not improved. We gave explicit consistency conditions on warped compactifications. These consistency conditions were derived from the requirement of having a globally well-defined solution to the equations of motion. If the conditions are violated the metric on the brane is not at a stationary point of the 4d effective action. Various examples with singularities are inconsistent unless the singularities are screened by additional sources. The vacuum energy at these sources needs to be fine-tuned.

Another issue appearing in the literature is that for $|b| = 4/3$ and vanishing bulk potential only solutions with zero cosmological constant exist [9, 20]. We pointed out that this advantage turns into a disadvantage as soon as a bulk potential for the scalar is switched on. Then only solutions with a non-zero cosmological constant exist (the numerical value is given by parameters of the bulk potential). Hence, in addition of fine-tuning $|b| = 4/3$ one needs to fine-tune parameters in the bulk potential even at the classical level.

Finally, we pointed out that quantum corrections due to the theory living on a brane may alter the functional form of the coupling of the scalar to the brane. Other open problems are the stabilization of the scalar and the inter-brane distance.

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